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Mathematical study of the solutions of a diffusion equation with exponential diffusion coefficient: explicit self-similar solutions to some boundary value problems

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Abstract. Explicit self-similar solutions are systematically obtained for a non-linear diffusion equation with a diffusion coefficient that depends exponentially on the transported magnitude ($D(u) = K_1 \, e^{\alpha u}$), for some boundary conditions associated with the parameter λ of the transformation group. Group elements under which the Hamiltonian of this physical system remains invariant enable us to find new magnitudes which remain invariant under transformation for each specific λ value in turn. This allows physical properties of the system to be established. The sign of α limits the possibility of obtaining self-similar solutions in any kind of diffusion process for those group elements belonging to the open interval $(0, \frac{1}{2})$; if $\alpha > 0$ then only self-similar solutions for absorption processes will be found and if $\alpha < 0$, only the cession processes will possess self-similar solutions. The value $\lambda = 0$ leads to an analytic solution to the problem with free boundaries (the Stefan problem).

1. Introduction

The study of non-linear transport phenomena in theoretical and experimental physics is generating much interest due to the many situations in Nature that must be described within this framework. An investigation of the general aspects of this type of problem is therefore very useful in order to apply the conclusions to individual cases.

This behaviour is well described using a non-linear diffusion coefficient in the corresponding transport equation. In this work an exponential dependent on the transport magnitude is chosen as it is the commonest situation in physical systems [1]. In order to obtain the information desired using this working hypothesis, the transport equation under consideration is written as

$$\partial u / \partial t = \operatorname{div}(K_1 \exp(\alpha u) \operatorname{grad} u) \qquad (1)$$

with u the transport variable, K_1 an arbitrary constant dependent on each individual case, and α the degree of non-linearity of the system. α was chosen as a real number due to the evidence of increasing and decreasing diffusion coefficients within the transport magnitude. We shall consider the usual one-dimensional frame; in this way the solutions of equations (1) are obtained more simply and it is then very easy to extend the results to more complex symmetries.

In § 2 we give an outline of the method used in this paper. Our philosophy is to study the problem thoroughly from an analytical point of view in order to simplify it, using the transformation group technique as far as possible, and turn to numerical methods if compelled to by the nature of the solution [2-5]. This allows us to transform a partial differential equation into an infinite family of ordinary differential equations which depend on only one parameter.

A functional transformation is applied to the transport equation to give us a greater number of solutions than if we had applied the stretching group technique directly to equation (1). This transformation is defined by

$$u^* = \int_{u_0}^{u} \frac{\exp(\alpha u)}{\exp(\alpha u_0)} \, du \tag{2}$$

where u_0 is a reference value of the transport magnitude. By substituting (2) into (1) we obtain a new partial differential equation in which the properties of the physical system are characterised by a diffusivity coefficient. In this way, the degree of non-linearity has been reduced, and the equation takes the form

$$\partial z/\partial t = Kz\, \partial^2 z/\partial x^2 \tag{3}$$

in which the diffusivity demonstrates a linear dependence on the new magnitude z, defined by

$$z = 1 + \alpha u^* \tag{4a}$$

$$K = K_1 \exp(\alpha u_0). \tag{4b}$$

The variable z contains information about the transport magnitude u as well as the value of α, which influences the evolution of the system.

Obviously, since we are dealing with energy, matter, etc, the transported magnitude $(u(x, t) \geqslant 0)$ will always be positive, leading to

$$\alpha > 0 \qquad z(x, t) \geqslant 1$$

$$\alpha < 0 \qquad z(x, t) \in [0, 1].$$

As a result of leaving each boundary condition invariant under the transformation group, a single parameter value is selected [1].

In § 3, we shall demonstrate that this type of solution represents the asymptotic state of a large set of initial conditions in the problem. In § 4 we obtain particular solutions for the set of boundary conditions associated with parameter values included in the closed interval [0, 1].

(A) Isolated semi-infinite system subjected to an energy pulse, matter or movement quantity at the initial instant.

(B) Semi-infinite system subjected to a constant flow of energy, matter, etc at the exterior contact boundary.

(C) Finite system subjected to a free boundary process.

New physical magnitudes which remain invariant under the transformation group are found when the physical system is submitted to a boundary condition associated with parameter values belonging to the open interval $(0, \frac{1}{2})$.

2. Transformation group technique

A finite group of transformations is chosen because it is easier to apply than the corresponding infinitesimal group, and the results are equivalent [1].

Let us define the following one-parameter group:

$$\bar{z} = a^p z \tag{5a}$$

$$\bar{x} = a^q x \tag{5b}$$

$$\bar{t} = a^m t \tag{5c}$$

where a is the parameter of the group and p, q and m are real constants yet to be determined. Now, equation (3) is required to be formally invariant under the application of (5a)-(5c). This invariance produces relations involving p, q and m parameters, and these determine a particular element of the group [6]. The invariance leads to the relation

$$p - 2q + m = 0. \tag{6}$$

For the sake of simplicity one can take $q = \lambda m$. In this way, the independent invariants of the group are

$$J = z(t/T)^{(1-2\lambda)} \tag{7a}$$

$$\xi = x(t/T)^{-\lambda} \tag{7b}$$

where either J or ξ, or any function of them, are invariants. T is a characteristic time in the evolution of the system and the origin of time should be taken conventionally at $t = T$. This is a suitable choice because for $t = T$ the invariants become $\xi = x$, $J = z$ and the two spaces coincide at the initial time.

In order to obtain invariant solutions [6], we assume

$$z(x, t) = (t/T)^{(2\lambda-1)} f(\xi) \tag{8}$$

$f(\xi)$ being an arbitrary function of the new variable ξ. If we define $y = KTJ$ and then substitute (7a) and (7b) into (3), we obtain the reduced equation

$$y\frac{d^2y}{d\xi^2} + \lambda\xi\frac{dy}{d\xi} - (2\lambda - 1)y = 0. \tag{9}$$

Briefly, the one-parameter group (5a)-(5c) has allowed us to transform a non-linear partial differential equation into an infinite family of quasilinear ordinary differential equations which are characterised by an arbitrary real parameter λ. In this way, if a solution $y = f(\xi)$ is found for equation (9) the corresponding solution to equation (1) is obtained by taking the inverse transformation

$$u = (1/\alpha)\log[(y/KT)(t/T)^{(2\lambda-1)}] \tag{10a}$$

$$x = \xi(t/T)^{\lambda}. \tag{10b}$$

Note that if the transformation group is applied directly to equation (1), a single invariant is obtained in (7a) and (7b), corresponding to the value $\lambda = \frac{1}{2}$, which generates only one ordinary differential equation, the solution of which has been already found by another method [7]. Therefore the number of solutions would be drastically reduced. Thus, using the present method, it is possible to obtain an infinite family of equations and the solution mentioned above is the member of this family corresponding to $\lambda = \frac{1}{2}$.

The parameter λ is determined when the temporal behaviour of some physical magnitudes is fixed [1]. Since these magnitudes should remain invariant under the transformation group, the solution will be compatible with the defined group. The method described for determining the value of the parameter λ can be generalised to an arbitrary and extensive set of boundary conditions, which accounts for a larger number of solutions by means of application of the transformation group.

3. Type of solutions

From the reduced equation (9) we can obtain a set of solutions which are singular, because part of the information contained in equation (1) was lost when we eliminated the time variable. Now the objective is to discover whether these singular solutions are never representative of regular solutions or, in contrast, are highly representative. For example, these solutions could tend asymptotically to the regular solution as t increases.

To tackle this problem, we are going to use another kind of transformation group [5, 8, 9], the so-called quasi-invariance group of transformations, where the requirements are weaker and there is no reduction of variables in the system. Thus the transformed equation explicitly contains all temporal or spatial modifications imposed on the system.

Let us define the following transformation group:

$$x = C(t)\xi \tag{11a}$$

$$t = \theta(t) \tag{11b}$$

$$z = A(t)\phi(\xi, \theta) \tag{11c}$$

where the characteristic parameters are time functions which produce a new family of partial differential equations, the coefficients of which are also temporal functions. By substituting (11a)-(11c) into (3) we obtain

$$K\phi\frac{\partial^2\phi}{\partial\xi^2} + \frac{C\dot{C}}{A}\xi\frac{\partial\phi}{\partial\xi} - \frac{\dot{A}C^2}{A^2}\phi = \frac{C^2}{A}\dot{\theta}\frac{\partial\phi}{\partial\theta} \tag{12}$$

where the dot indicates a time derivative.

Now, in order to determine the most convenient elements of the transformation group, we impose two requirements concerning the system evolution in the new space (ϕ, ξ, θ):

(i) the system, described in the space ϕ, ξ, θ, must conserve the Hamiltonian formalism;

(ii) the new temporal scale must be taken so that the system reaches its asymptotic state in a finite time.

The first assumption fixes the value of $A(t)$ and $C(t)$ in the form

$$C(t) = (t/T)^{\lambda} \tag{13a}$$

$$A(t) = (t/T)^{(2\lambda-1)}. \tag{13b}$$

The second condition is verified when the coefficient of the right-hand side of (12) goes to zero as t goes to infinity. In this way, the $\theta(t)$ function must satisfy

$$C^2\dot{\theta}/A = (t/T)^{-\gamma} \qquad \gamma > 0 \tag{14}$$

hence

$$\theta(t) = (1/\gamma)[1-(t/T)^{-\gamma}] \qquad \gamma > 0. \tag{15}$$

Consequently, it may be seen that the evolution interval in the new space is $[0, 1/\gamma]$, whereas the corresponding real time is $[T, \infty]$. γ is chosen on the condition that the coefficient of $\partial\phi/\partial\theta$ in (12) goes to zero and it is possible irrespective of the asymptotic behaviour of the system in the new space. In this way, by studying the solutions in the neighbourhood of $\theta_{\lim} = 1/\gamma$, the information obtained represents the asymptotic state of the system in the original space. By examining (12) in the light of the above results, we find a new partial differential equation

$$(t/T)^{-\gamma}\frac{\partial\phi}{\partial\theta}=\phi\frac{\partial^2\phi}{\partial\xi^2}+\lambda\xi\frac{\partial\phi}{\partial\xi}-(2\lambda-1)\phi \qquad \gamma>0 \tag{16}$$

which is of the same type as (9); in fact they are coincident as $t\to\infty$ because for physical situations $\partial\phi/\partial\theta$ does not grow to infinity, which enables us to require that the particular solutions obtained from (9) are the asymptotic solutions of (12) whatever the initial conditions.

4. Self-similar solutions associated with some boundary conditions

We shall now study particular solutions of the non-linear equation (1) with the following associated boundary conditions in semi-infinite and infinite media.

(A) *Semi-infinite medium.* The total transfer of energy, matter, etc to the physical system through the contact frontier from the exterior medium is carried out by a source plane at the initial instant:

$$Q_0=\int_0^\infty u\,\mathrm{d}x=cte \qquad \forall t. \tag{17}$$

The spatial distribution of the transported magnitude is considered to satisfy the conservation laws. This condition allows us to suppose that, at the initial instant, all energy, matter, etc is concentrated on the border. This can be expressed mathematically by

$$u(x,0)=Q_0\delta(x) \tag{18}$$

where $\delta(x)$ is the Dirac distribution function.

(B) *Semi-infinite medium.* The flow of energy, matter, etc between the physical system and the exterior medium through the contact frontier is constant at all times:

$$F_0=-K_1\exp(\alpha u(0,t))\left.\frac{\partial u}{\partial x}\right|_{x=0}=cte \qquad \forall t. \tag{19}$$

(C) *Finite medium.* A flow of energy is supplied to a finite medium with a free boundary

$$u(x_i(t),t)=u_0 \qquad \forall t. \tag{20}$$

4.1. Self-similar solutions associated with boundary condition A

Equation (1) with the boundary condition (17) is rewritten in the form

$$\frac{\mathrm{d}Q_0}{\mathrm{d}t}=\int_0^\infty\frac{\partial u}{\partial t}\,\mathrm{d}x=\left[K_1\exp(\alpha u)\frac{\partial u}{\partial x}\right]_0^\infty=0 \qquad \forall t \tag{21}$$

which allows us to specify the way in which the gradient function behaves at extremes of the system. If the system remains isolated after application of the perturbation then

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=0. \tag{22}$$

This condition also requires that the gradient function be equal to zero at infinity in order to satisfy relation (21), and so

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}=\left.\frac{\partial u}{\partial x}\right|_{x=\infty}=0. \tag{23}$$

In the new space that defines the transformation group, the variable $y(\xi)$ should satisfy the following boundary conditions:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}\xi}\right|_{\xi=0}=\left.\frac{\mathrm{d}y}{\mathrm{d}\xi}\right|_{\xi=\infty}=0 \tag{24}$$

as can be deduced from applying the independent invariants of the transformation group and the functional transformation to (23).

We must find those values of λ which lead to differential equations whose solutions satisfy the conditions imposed by (24). As we shall show, the set of equations characterised by values of λ in the open interval $(0, \frac{1}{2})$ provides a solution that satisfies the zero gradient function condition at the extremes of the system. If $0<\lambda<\frac{1}{2}$, the family of equations (9) can be rewritten as

$$y\,\mathrm{d}^2y/\mathrm{d}\xi^2+\lambda\xi\,\mathrm{d}y/\mathrm{d}\xi+|2\lambda-1|y=0. \tag{25}$$

Since $y(\xi)$ is defined positive, equation (25) requires the defined solution to be strictly decreasing since the first- and second-order derivatives cannot both be simultaneously positive.

The boundary condition (24) requires that the gradient function at $\xi=0$ be zero, and imposes an upper limit on the solution function at that point, since

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi=0}=-|2\lambda-1|. \tag{26}$$

When the solution function decreases at points close to the origin the first derivative will take negative values, inducing the second derivative to tend toward zero in accordance with equation (25). There are two ways in which the second derivative may approach zero:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi_i}=0 \qquad \text{and } y(\xi_i)\neq 0$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi_i}=0 \qquad \text{and } y(\xi_i)=0.$$

Let us now study each of these options in turn.

The option $y(\xi_i)\neq 0$ can present the following possibilities:

Case (a) $\left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi_i}=0$ when $\xi_i\neq\infty$

Case (b) $\left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi_i}=0$ when $\xi_i=\infty$

Case (c) $\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}$ never reaches zero.

If case (a) occurs and the spatial second derivative of the function reaches zero at a finite distance from the origin, the function gradient takes the value

$$\left.\frac{\mathrm{d}y}{\mathrm{d}\xi}\right|_{\xi_i} = -\frac{|2\lambda-1|}{\lambda}\frac{y_i}{\xi_i} \tag{27}$$

enabling us to determine how the second derivative evolves at subsequent points:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi_{i+1}} = |2\lambda-1|\left(\frac{\xi_i+\Delta\xi}{\xi_i}\frac{y_i}{y_{i+1}}-1\right)>0. \tag{28}$$

Positive values for the second derivative function induce a decrease in the negative growth of the gradient and make it tend toward zero. If we suppose that the gradient reaches zero at $\xi_j > \xi_i$ then, since the second derivative remains positive for reasons of continuity, this position would correspond to a minimum for the solution function. There are only two possibilities for the behaviour of the solution function beyond ξ_j: to grow or to stay at its minimum value. The first option is incompatible with differential equation (25) so, once the solution function reaches its minimum it remains there. Therefore, if the spatial derivative of the function is zero at $\xi = \xi_j$, all points $\xi > \xi_j$ are not reached by the perturbation and it satisfies the requirement

$$\left.\frac{\mathrm{d}y}{\mathrm{d}\xi}\right|_{\xi=\infty} = 0.$$

If case (b) occurs, the spatial second derivative of the function reaches zero at infinity so it must be verified that

$$\lim_{\xi\to\infty}(\lambda\xi\,\mathrm{d}y/\mathrm{d}\xi+|2\lambda-1|y)=0. \tag{29}$$

Therefore, in order to maintain the physical criteria, the limits

$$\lim_{\xi\to\infty}(\lambda\xi\,\mathrm{d}y/\mathrm{d}\xi)=0 \tag{30}$$

$$\lim_{\xi\to\infty}(|2\lambda-1|y)=0 \tag{31}$$

cause both the function $y(\xi)$ and its gradient to become zero at $\xi =$ infinity. Once more our requirement of a zero gradient at infinity is satisfied.

The last possibility, case (c), that we noted for the behaviour of the spatial second derivative of the function carries us to a nonsense. If equation (25) must be satisfied, at least at infinity, the function and all its derivatives must be zero.

We now examine the second option, namely

$$y(\xi_i)=0 \qquad \text{when} \qquad \left.\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}\right|_{\xi_i}=0.$$

Under these conditions, and according to (25), we must comply with

$$\lambda\xi_i\left.\frac{\mathrm{d}y}{\mathrm{d}\xi}\right|_{\xi_i}=0. \tag{32}$$

Since $\xi_i \neq 0$, the spatial derivative of the function at $\xi = \xi_i$ must be zero and, as we have already found, once the minimum is reached, the function remains at that minimum.

This superficial analysis has allowed us to find the set of differential equations which generates the solutions corresponding to the boundary condition (24). Any one of them fulfils the requirement of maintaining the quantity Q_0 constant in real space. This set of equations has the peculiarity that each value of λ leaves a new physical magnitude invariant. In order to select the adequate parameter and so fix the particular asymptotic solution determining the evolution of the diffusion process we are studying, it is necessary to analyse the behaviour of these new magnitudes in the physical system.

Next, some physical magnitudes that remain invariant under the transformation are detailed [10], depending on the value assigned to the parameter, within the considered interval.

(i) When $\lambda = \frac{1}{3}$ the physical magnitude defining the integral function,

$$M = \int_0^\infty \exp(\alpha u)\, \mathrm{d}x = cte \qquad \forall t \tag{33}$$

remains invariant under this element of the transformation group. Effectively, and since $y\, \mathrm{d}\xi = cte$ at all times,

$$M = \frac{1}{KT}(t/T)^{(3\lambda-1)} \int_0^\infty y\, \mathrm{d}\xi = cte \qquad \forall t \tag{34}$$

when $\lambda = \frac{1}{3}$.

If a physical system satisfies condition (33) and is submitted to the boundary condition we are examining, the evolution of the perturbation is determined by

$$y\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \tfrac{1}{3}\xi\frac{\mathrm{d}y}{\mathrm{d}\xi} + \tfrac{1}{3}y = 0. \tag{35}$$

Equation (35) has no class C solution, so it is solved by numerical calculations.

(ii) When $\lambda = \frac{1}{4}$ the physical magnitude defining the integral function,

$$N = \int_0^\infty x \exp(\alpha u)\, \mathrm{d}x = cte \qquad \forall t \tag{36}$$

remains invariant under this element of the transformation group. Effectively, since $\xi y\, \mathrm{d}\xi = cte$ at all times,

$$N = \frac{1}{KT}(t/T)^{(4\lambda-1)} \int_0^\infty \xi y\, \mathrm{d}\xi = cte \qquad \forall t \tag{37}$$

when $\lambda = \frac{1}{4}$.

The study of the behaviour of the physical system against this magnitude is carried out by analysing its temporal variation, because of the divergence presented by the function (36). The evolution of the perturbation is determined by

$$y\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \tfrac{1}{4}\xi\frac{\mathrm{d}y}{\mathrm{d}\xi} + \tfrac{1}{2}y = 0. \tag{38}$$

Equation (38) is solved by numerical calculations. Figure 1 shows the numerical solutions generated by equations (35) and (38), and so displays the zero value for the gradient at the extremes of the system, which is a necessary condition for solution under the boundary condition A.

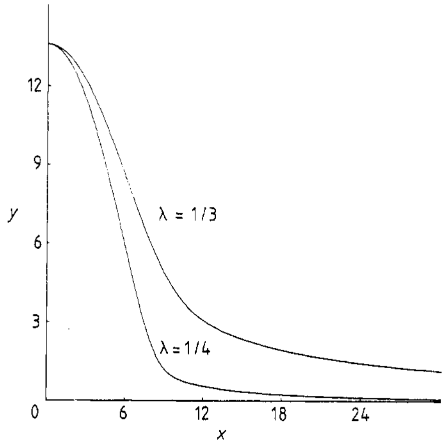

Figure 1. Numerically calculated solutions of equations (35) and (38). Both solutions manifestly satisfy the zero gradient condition at the boundary of the physical system (boundary condition A).

As we are interested in finding the solutions in (u, x, t) space, we shall study the influence of the parameter α on the above solutions, through the gradient function of the transported magnitude

$$\frac{\partial u}{\partial x} = \frac{1}{\alpha y}(t/T)^{-\lambda}\frac{\mathrm{d}y}{\mathrm{d}\xi} \tag{39}$$

since the spatial derivative of the function is always negative or zero as a result of equation (25); the type of diffusion process under study is limited by the value of α. Self-similar solutions can only be obtained under the following conditions:

$\alpha > 0$ absorption processes, since $\partial u/\partial x < 0$ always

$\alpha < 0$ cession processes, since $\partial u/\partial x > 0$ always.

As illustrative examples, we show, in figures 2 and 3, the temporal evolution of the transported magnitude in an absorption process in a medium characterised by $\alpha > 0$ which also satisfies the conditions (33) and (36), respectively.

4.2. *Self-similar solutions associated with boundary condition B*

The boundary condition (19) is rewritten in the transformed space:

$$F_0 = -\frac{1}{\alpha T}(t/T)^{(\lambda-1)}\frac{\mathrm{d}y}{\mathrm{d}\xi}\bigg|_{\xi=0} = cte \qquad \forall t. \tag{40}$$

This condition must remain invariant, which requires the parameter λ to take the value 1. The reduced equation takes the form

$$y\frac{\mathrm{d}^2 y}{\mathrm{d}\xi^2} + \xi\frac{\mathrm{d}y}{\mathrm{d}\xi} - y = 0. \tag{41}$$

Since equation (41) has no class C solutions, numerical methods are used to solve it.

It is interesting to note that the same physical system does not behave symmetrically when faced with absorption or cession processes of flow phenomena.

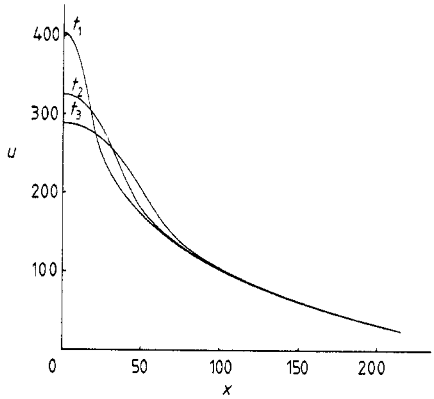

Figure 2. Temporal evolution for a diffusion process subjected to boundary condition A when the medium is characterised by $\alpha > 0$ and maintains the magnitude defined by (33) invariant through time ($M = \int_0^\infty \exp(\alpha u)\,\mathrm{d}x$; $t_1 = T$, $t_2 = 10T$, $t_3 = 30T$; $\lambda = \frac{1}{3}$, $\alpha = 0{\cdot}01$, $1/KT = 1$).

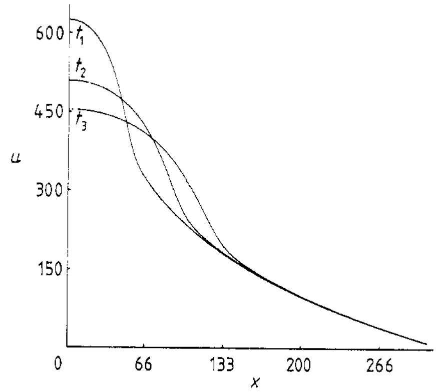

Figure 3. Temporal evolution of a diffusion process subjected to boundary condition A in a medium characterised by $\alpha > 0$. The medium maintains the magnitude defined by (36) invariant through time ($N = \int_0^\infty x \exp(\alpha u)\,\mathrm{d}x$; $t_1 = T$, $t_2 = 10T$, $t_3 = 30T$; $\lambda = \frac{1}{4}$, $\alpha = 0{\cdot}01$, $1/KT = 1$).

When the spatial derivative of the function is less than zero at the origin a peculiar solution is generated, because of the sign change presented by the gradient at points close to the origin. The solution for equation (41) manifests an extremely non-linear behaviour in the medium. The region close to the origin absorbs or cedes flow proceeding from either the system or its neighbours depending on whether $\alpha > 0$ or $\alpha < 0$, respectively. Figure 4 presents the temporal evolution of a physical system characterised by $\alpha < 0$.

If the spatial derivative of the function is greater than zero at the origin (figure 5), an infinite flow is required at the border and a finite flow in those points that are

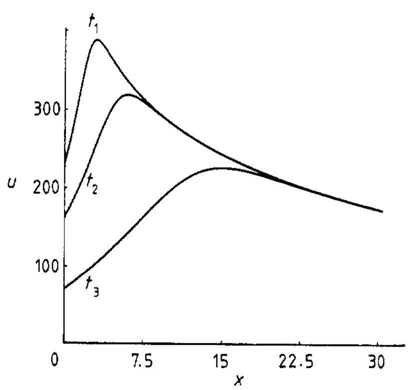

Figure 4. A medium characterised by $\alpha < 0$ when subjected to a diffusion process with constant flow cession at the border (boundary condition B) presents highly non-linear behaviour. The system does not evolve beyond the position predetermined by the boundary condition ($t_1 = T$, $t_2 = 2T$, $t_3 = 5T$; $\lambda = 1$, $\alpha = -0{\cdot}01$, $1/KT = 0{\cdot}01$).

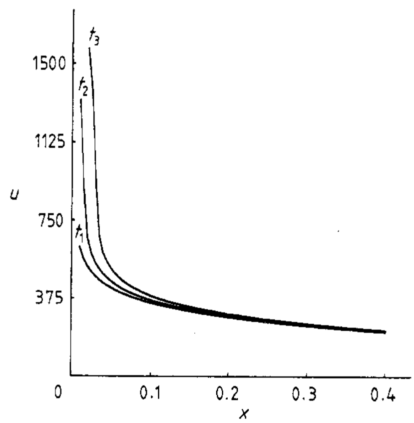

Figure 5. The temporal evolution of a diffusion process in a medium characterised by $\alpha < 0$ when it is subjected to a constant flow absorption process at the border (boundary condition B). Threshold energy is required to initiate evolution ($t_1 = T$, $t_2 = 5T$, $t_3 = 10T$; $\lambda = 1$, $\alpha = -0{\cdot}01$, $1/KT = 1$).

infinitely close to the border, so that the system is perturbed, as can be deduced by studying equation (41).

In both situations, the physical system presents the same behaviour: the medium does not evolve beyond a position that is predetermined by the flow boundary condition.

4.3. *Self-similar solution associated with boundary condition C*

The free boundary problem in a medium with a linear behaviour was studied by Bluman [4]. We solve the same problem when the medium presents the type of non-linearity that we study here. The boundaries of the finite medium are defined at the initial instant by

$$x_1(t=T)=0 \tag{42a}$$

$$x_2(t=T)=L \tag{42b}$$

when the domain of the spatial variable is defined by

$$x_1(t)<x<x_2(t) \tag{43}$$

and the boundary condition that should satisfy equation (1) is

$$u(x_i(t),t)=u_0 \qquad \forall t. \tag{44}$$

Once the solution generated by conditions (42*a*), (42*b*) and (44) is known, the flow needed to develop the diffusion process is also known, since it depends on the penetration velocity of the border. Making use of the independent invariants (7*a*) and (7*b*), the boundary conditions are rewritten

$$x=\xi(t/T)^{\lambda}. \tag{45}$$

The imposition of invariance on the free boundary selects zero as the value for λ and the reduced equation takes the form

$$y\frac{d^2y}{d\xi^2}+y=0. \tag{46}$$

Equation (46) has a class C solution, the trivial solution $y(\xi)=0$ and the general one

$$y(\xi)=\tfrac{1}{2}(a+b\xi-\xi^2). \tag{47}$$

Border evolution is described by

$$x(t)=\tfrac{1}{2}\{b\pm[b^2+4(a-2Kt)]^{1/2}\}. \tag{48}$$

Equation (48) permits us to find the propagation velocity for the borders

$$v(t)=2K/(b-2x(t)) \tag{49}$$

and the duration of the diffusion process

$$t=(b^2+4a)/8K. \tag{50}$$

The influence of the parameter α on these solutions can be studied from the temporal variation for the transported magnitude

$$\frac{\partial u}{\partial t}=-(1/\alpha t) \qquad \forall x. \tag{51}$$

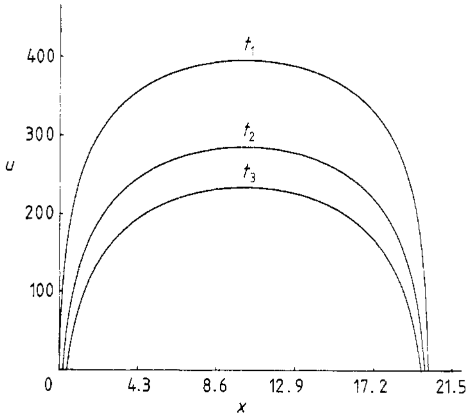

Figure 6. Temporal evolution of a process with free boundaries (boundary condition C) in a finite medium characterised by $\alpha > 0$. The system cedes energy until reaching equilibrium with the external medium ($t_1 = T$, $t_2 = 3T$, $t_3 = 5T$; $\lambda = 0$, $\alpha = 0{\cdot}01$, $1/KT = 1$).

This result restricts the application of the method to the following situations:

(i) if $\alpha < 0$ then $\partial u/\partial t > 0$ and only absorption processes possess self-similar solutions;

(ii) if $\alpha > 0$ then $\partial u/\partial t < 0$ and only cession processes possess self-similar solutions.

Figure 6 shows the temporal evolution of a physical system characterised by $\alpha > 0$ in an energy cession process.

5. Conclusions

This method reduces the mathematical complexity of the non-linear problem, and makes it possible to apply numerical methods so that instability and other inherent problems are avoided. Both the functional transformation on the problem equation and the subsequent formal invariance under one-parameter transformation groups allow us to define a set of new variables in the physical system, which enable us to describe this evolution, through a family of non-linear ordinary differential equations. On the other hand, the choice of adequate variables (independent invariants of the group) in each case is determined by boundary conditions and this fixes solutions unambiguously.

Although, in principle, the reduction of the number of independent variables in the equation implies a loss of information, the solutions derived from the reduced equation are representative of the asymptotic behaviour of the physical system under any perturbation of an extensive set of initial conditions which corresponds to the same boundary condition.

Self-similar solutions for boundary conditions associated with parameter values belonging to the closed interval $[0, 1]$ have been obtained. This interval generates a set of differential equations that produce convergent solutions, i.e. they define evolutions of the physical system that tend to reach a perturbed equilibrium.

The elements of the transformation group that corresponds to values of λ in the open interval $(0, \frac{1}{2})$ present the following properties when they are applied to equation (3): any element of the group leaves invariant the boundary condition defined by (17), while each particular value leaves different physical magnitudes invariant, like those defined by (33) and (36). A study of the behaviour of the system against these magnitudes permits the adequate differential equation to be fixed. This adequate differential equation generates the solutions that correspond to the boundary condition supporting the medium.

The sign of α limits the possibility of studying any diffusion process in the open interval $(0, \frac{1}{2})$:

if $\alpha > 0$ absorption processes

if $\alpha < 0$ cession processes.

The problem of a free boundary in a finite medium, which corresponds to the parameter λ being zero, is completely studied since the resulting differential equation possesses a class C solution and therefore it allows us to find the position and velocity of the border, and the duration and energy needed for evolution of the process. These solutions are of great interest for the food industry in the manufacture of solid elements and in freezing.

The fact that a system does not evolve beyond a position that is predetermined by a constant energy flow at the border determines the exponential non-linearity for the diffusion coefficient of the medium. This property has been corroborated by the analysis of experimental data [1, 11].

We remark that some media that are non-linear with regard to diffusion phenomena, and which are normally treated by the addition of new phenomenological terms to the linear model, can be analysed by modelling a non-linear equation with this type of diffusion coefficient [1, 11].

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